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Spatial Landau damping in plasmas with three-dimensional κ distributions

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The increase in linear Landau damping in κ-distributed plasmas compared to thermal equilibrium plasmas is studied by solving a boundary value problem for the spatially damped plasma waves generated by a planar grid electrode with an applied time harmonic potential. Solutions are computed for the plasma potential versus the distance from the electrode for different values of the parameter κ (kappa). The velocity parameter v_0 of the distribution function is chosen so that, as the parameter κ varies, the kinetic temperature of the plasma remains constant. The exact solutions of this problem are also compared to approximate solutions derived from the theory of normal modes, that is, from the roots of the dispersion relation. This model problem demonstrates the significant increase in Landau damping by electrons which occurs for small values of the parameter κ. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1885474]

I. INTRODUCTION

In space plasmas, ion and electron distribution functions are usually observed to contain power law tails. This was first discovered during early measurements of solar wind electrons where κ distributions (defined in Sec. III) were found to give a reasonable fit to the data.[1,2] Linear Landau damping by electrons is greater in plasmas having particle distribution functions with power law tails than in plasmas at thermal equilibrium because there are more resonant particles available at higher energies to participate in the damping process. The magnitude of this effect is of interest in space physics where κ distributions are common and where collisionless processes, such as Landau damping and other wave-particle interactions, provide the primary means of plasma relaxation and plasma energization. The purpose of this paper is to compare the magnitude of Landau damping in κ and Maxwell distributed plasmas using a model problem for which exact solutions can be analyzed in detail. Only Landau damping by electrons shall be considered.

The problem is to compute the spatial damping of electrostatic plasma waves generated by a plane electrode driven by a time harmonic potential. Here, one is concerned with spatial damping (damping in space) rather than temporal damping (damping in time). As is well known, the damping rates for the two types of damping are determined by the roots of the plasma dispersion relation $D(k,\omega)=0$ where $D(k,\omega)$ is the plasma dielectric function.[3] The temporal damping problem is given the most attention in the literature, probably for historical reasons. But both types of damping are important in applications.

In a spatially homogeneous plasma, the normal modes of oscillation are characterized by a frequency and wave number determined by the roots of the dispersion relation $D(k,\omega)=0$. There are two separate classes of normal modes, temporally damped normal modes,

$$\phi_k(x,t) = A(k)\exp\{i[kx-\omega(k)t]\}, \tag{1}$$

which are useful for the solution of initial value problems, and spatially damped normal modes,

$$\phi_\omega(x,t) = A(\omega)\exp\{i[k(\omega)x-\omega t]\}, \tag{2}$$

which are useful for the solution of boundary value problems. In Eq. (1), the wave number k is any nonzero real number and the complex angular frequency $\omega(k)$ is determined by the plasma dispersion relation. In Eq. (2), the frequency ω is any nonzero real number and the complex wave number $k(\omega)$ is determined by the plasma dispersion relation.

For a planar electrode immersed in a plasma in the plane $x=0$ and driven with an applied voltage $V(t)=\phi_0\sin(\omega t)$, the solution for the electrostatic potential in the plasma is expected to take the form

$$\phi(x,t) = -\phi_0 \sin(k_r|x|-\omega t)\exp(-k_i|x|), \tag{3}$$

where $k=k_r+ik_i$ is the unique root of the dispersion relation with the smallest imaginary part (if it exists), k_r and k_i are both real and positive, and ϕ_0 is a constant. If a unique root does not exist, as in the case of the Maxwell distribution, then the "dominant" root of the dispersion relation must be used instead (see Sec. IX). Equation (3) is the solution suggested by the theory of normal modes. Of course, the exact solution depends on the physical boundary conditions. In this paper, one possible set of boundary conditions is used to develop an exact solution of the linearized Vlasov equation and the result is compared to the normal mode solution (3).

The significant increase of Landau damping in κ-distributed plasmas was first studied in the pioneering papers of Thorne and Summers,[4] and Summers and Thorne.[5] They solved the plasma dispersion relation for a κ-distributed, unmagnetized plasma, and computed the damping rate as a function of the real wave number k for different values of κ (kappa). The results were then compared to the thermal equilibrium values. Analysis of collisionless damping rates and the growth rates of various instabilities in both magnetized and unmagnetized plasmas have

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since been carried out for various hybrid combinations of κ and Maxwell distribution functions. Because it is impractical to mention all of the many references, just a few shall be cited here.[6–11] The work described in Refs. 6–11 and in many related papers is important because of its wide range of applications in space physics.

The outline of this paper is as follows. The boundary value problem for the dynamic shielding of a time harmonic sheet charge is derived in Sec. II and its relationship to the problem of an electrode with an oscillating potential is described. Section III provides the definition of the κ distribution function and its statistical moments. In Sec. IV, the plasma dispersion function for the κ distribution is derived in such a way that the plasma dielectric function takes the same form as for the Maxwell distribution. The plasma dispersion relation for spatial damping is described in Sec. V. In Sec. VI, the solutions for static Debye shielding are briefly mentioned. In Sec. VII, the solution of the dynamic shielding problem is decomposed into near-field and far-field components and the solution is thereby obtained for an electrode with an oscillating potential. In Sec. VIII, the exact solution for the plasma potential is computed for different values of κ and, in Sec. IX, the results are compared to the normal mode solution (3).

II. BOUNDARY VALUE PROBLEM FOR DYNAMIC DEBYE SHIELDING

The plasma waves generated by an electrode with an oscillating potential can be obtained from the solution of a related problem. Consider the dynamic screening due to a sheet charge in the plane $x=0$ with a time harmonic charge density of the form $\rho(x,t)=\rho_0\delta(x)\cos(\omega t)$, where ρ_0 is the surface charge density (a constant) and $\delta(x)$ is the Dirac delta function. It will be shown that the solution of this problem contains two components, a far-field electric field of the form

$$E_F(x,t) = \frac{\rho_0}{2\epsilon_0[1-(\omega_p^2/\omega^2)]}\,\mathrm{sgn}(x)\cos(\omega t), \tag{4}$$

where $\mathrm{sgn}(x)$ is the algebraic sign of x, plus a near-field component consisting of spatially damped plasma waves that propagate away from the source. The near-field component is a solution of the linearized Vlasov equations with the property that the potential approaches $\phi_0\sin(\omega t)$ as $x\to 0$ and vanishes as $|x|\to\infty$. Thus, this is the solution of interest, that is, this is the field generated by an electrode with an oscillating potential. The complete solution is derived as follows.

By symmetry, $\partial/\partial y=\partial/\partial z=0$. Thus, the waves will propagate along the x direction. At high frequencies, the motion of the ions is negligible due to their much larger mass. The ions simply provide a uniform background of positive charge which preserves overall charge neutrality. Assuming that the magnitude of the forcing is sufficiently small, the response of the plasma is governed by the linearized Vlasov equations

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{eE}{m}\frac{\partial f_0}{\partial v} = 0, \tag{5}$$

$$\frac{\partial E}{\partial x} = -\frac{n_0 e}{\epsilon_0}\int f dv + \frac{\rho_0}{\epsilon_0}\delta(x)\cos(\omega t), \tag{6}$$

where f and E are the perturbed electron distribution function and electric field, respectively. The variable v is the x component of the velocity, that is, v_x, and the dependence on the velocity components v_y and v_z has been integrated out. The full distribution function is $n_0[f_0(v)+f(x,v,t)]$ where n_0 is the equilibrium density and $\int f_0 dv=1$. In Eqs. (5) and (6), $e>0$ is the electron charge, m is the electron mass, E is the x component of the electric field, ϵ_0 is the permittivity of free space, ρ_0 is the surface charge density, $\delta(x)$ is the delta function, and $\omega\geqslant 0$ is the driving frequency. Standard SI units are used throughout.

The right-hand side of Eq. (6) consists of two terms, the first representing the charge density of the plasma and, the second, the "externally imposed" charge density. The source term is an oscillating sheet charge with a charge density $\rho_0\cos(\omega t)$ which acts as a field generator or generator of plasma waves. For a static charge distribution $\omega=0$ and, as indicated below, the solution for the field is proportional to $\exp(-|x|/\lambda)$ where λ is the Debye length. Thus, the Eqs. (5) and (6) include the special case of static Debye shielding.

As first shown by Landau,[12] the physically correct solution of Eqs. (5) and (6) is obtained by applying the Fourier transform in space and the Laplace transform in time. Only the steady state solution is of interest here, that is, the time asymptotic solution as $t\to\infty$. This can be derived from the full Fourier and Laplace transformed solutions as shown, for example, by Krall and Trivelpiece[3] and in other well-known textbooks.[13,14]

Alternatively, the result for the steady state solution can be obtained directly from Eqs. (5) and (6) by assuming a harmonic time dependence of the form $\exp(-i\omega t)$ where ω contains a positive imaginary part (vanishingly small) which is allowed to approach zero at the end of the calculation. This approach ensures causality. Thus, the steady state solution is written as

$$E(x,t) = \mathrm{Re}[\tilde{E}(x,\omega)e^{-i\omega t}], \tag{7}$$

where $\tilde{E}$ is the complex amplitude. Hereafter, a tilde over any quantity denotes its complex amplitude. Note that the in-phase and quadrature components of $E(x,t)$, that is, the components proportional to $\cos(\omega t)$ and $\sin(\omega t)$ are given by the real and imaginary parts of $\tilde{E}(x,\omega)$, respectively.

Adopting the ansatz (7), together with a similar ansatz for $f(x,v,t)$, and then taking the Fourier transform (with respect to x) of Eqs. (5) and (6), one finds

$$(-i\omega + ikv)\tilde{f}(k,v,\omega) = \frac{e}{m}\tilde{E}(k,\omega)\frac{\partial f_0}{\partial v}, \tag{8}$$

$$ik\tilde{E} = -\frac{n_0 e}{\epsilon_0}\int \tilde{f} dv + \frac{\rho_0}{\epsilon_0}. \tag{9}$$

Solving Eq. (8) for $\tilde{f}$ and then substituting the result into Eq. (9), one finds the solutions

$$\tilde{f}(k,v,\omega) = \left(\frac{e}{ikm}\right)\frac{f_0'(v)}{v-(\omega/k)}\tilde{E}(k,\omega), \tag{10}$$

$$\tilde{E}(k,\omega) = \frac{\rho_0}{ik\epsilon_0 D(k,\omega)}, \tag{11}$$

where $f_0'(v)=\partial f_0/\partial v$ is the derivative of $f_0(v)$ with respect to v and $D(k,\omega)$ is the plasma dielectric function. The plasma dielectric function is defined by

$$D(k,\omega) = 1 - \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty}\frac{f_0'(v)}{v-(\omega/k)}dv, \quad \mathrm{Im}(\omega) > 0, \tag{12}$$

where $\omega_p^2 = n_0 e^2/\epsilon_0 m_e$ is the electron plasma frequency. After an integration by parts, the plasma dielectric function can be written in the equivalent form

$$D(k,\omega) = 1 - \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty}\frac{f_0(v)}{(v-\omega/k)^2}dv, \quad \mathrm{Im}(\omega) > 0, \tag{13}$$

or, making the change of variables $v' = kv/|k|$,

$$D(k,\omega) = 1 - \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty}\frac{f_0(kv/|k|)}{(v-\omega/|k|)^2}dv, \quad \mathrm{Im}(\omega) > 0. \tag{14}$$

This last change of variables is permitted because the Fourier transform variable k is real, that is, $\mathrm{Im}(k)=0$. Equation (14) shows that if $f_0(v)$ is an even function of v, that is, there is no equilibrium current flowing in the plasma, then $D(k,\omega)$ is an even function of k. This is the case for all the plasmas studied here.

The fact that $D(k,\omega)$ is an even function of k implies that the inverse Fourier transform of Eq. (11) can be written as

$$\tilde{E}(x,\omega) = \frac{\rho_0}{\pi\epsilon_0}\int_0^{\infty}\frac{\sin(kx)}{kD(k,\omega)}dk. \tag{15}$$

This is the solution of the dynamic screening problem first obtained by Buckley[15] in 1968. The keen observer may note that Eq. (11) contains a pole at $k=0$ while Eq. (15) does not. To see this, note that $D(k,\omega)$ is a continuous function of k as k approaches zero along the real axis and that $D(0,\omega)$ is nonzero [see Eq. (37)]. Therefore, the integrand in Eq. (15) is continuous at $k=0$. If the inverse Fourier transform of Eq. (11) is written as

$$\tilde{E}(x,\omega) = \frac{\rho_0}{2\pi\epsilon_0}\left[\int_{-\infty}^{\infty}\frac{\cos(kx)}{ikD(k)}dk + i\int_{-\infty}^{\infty}\frac{\sin(kx)}{ikD(k)}dk\right], \tag{16}$$

then the first integral on the right-hand side contains the pole and the second integral, which is nonsingular, yields Eq. (15). The first integral on the right-hand side of Eq. (16) can be defined as the Cauchy principal value, in which case it equals zero by symmetry. A mathematically rigorous proof of Eq. (15) must be based on the theory of distributions because the source term in Eq. (6) contains a delta function. This is discussed further in the Appendix.

The next step in the solution of the physics problem at hand is to separate the electric field (15) into near-field and far-field components. First, however, it is necessary to compute the plasma dispersion function.

III. THREE-DIMENSIONAL κ DISTRIBUTION

In astrophysics and space physics, the isotropic κ distribution in three dimensions is usually defined by

$$f_\kappa(\boldsymbol{v}) = A_\kappa\left(1+\frac{v^2}{\kappa v_0^2}\right)^{-(\kappa+1)}, \quad \kappa > 1/2, \tag{17}$$

where $\boldsymbol{v}=(v_x,v_y,v_z)$ is the velocity vector, $v=(v_x^2+v_y^2+v_z^2)^{1/2}$, and v_0 is some characteristic velocity.[1,4] The normalization constant is

$$A_\kappa = \frac{1}{(\pi v_0^2)^{3/2}\sqrt{\kappa}}\cdot\frac{\Gamma(\kappa)}{\Gamma(\kappa-1/2)} \tag{18}$$

and is chosen such that

$$4\pi\int_0^{\infty} f_\kappa(\boldsymbol{v})v^2 dv = 1. \tag{19}$$

The condition $\kappa > 1/2$ is necessary for this integral to converge. The κ distribution is closely related to the β distribution. In fact, the β function arises when computing the normalization factor A_κ and the statistical moments of f_κ. In the limit as $\kappa\to\infty$, the κ distribution approaches the well-known Maxwell distribution e^{-v^2/v_0^2}, including the correct normalization.

The velocity moments of the κ distribution are given by

$$\langle v^n\rangle = \frac{2}{\sqrt{\pi}}(\kappa v_0^2)^{n/2}\frac{\Gamma\left(\frac{n+3}{2}\right)\Gamma\left(\kappa-\frac{n+1}{2}\right)}{\Gamma\left(\kappa-\frac{1}{2}\right)}, \tag{20}$$

where n is an integer and $0\leqslant n\leqslant 2(\kappa-1)$. For an arbitrary real power $n\geqslant 0$, $\langle v^n\rangle$ is finite if and only if $n<2\kappa-1$. In applications, the kinetic temperature of an isotropic κ distribution is defined by the relation

$$\frac{1}{2}m\langle v^2\rangle = \frac{3}{2}k_BT, \tag{21}$$

which, using Eq. (20), implies

$$v_0^2 = \left(\frac{2\kappa-3}{\kappa}\right)\frac{k_BT}{m}, \tag{22}$$

where k_B is Boltzmann's constant. The second moment $\langle v^2\rangle$ is finite if and only if $\kappa > 3/2$. The family of all κ distributions at the same absolute temperature T is obtained by substituting $v_0^2=v_0^2(\kappa)$ from Eq. (22) into Eq. (17). Only distributions with $\kappa > 3/2$ are allowed.

IV. PLASMA DISPERSION FUNCTION

Integrating the three-dimensional κ distribution (17) over v_y and v_z, one finds the reduced distribution function

TABLE I. Plasma dispersion function and dielectric function for the first few integer values of κ. For an isothermal family of κ distributions, $\kappa > 3/2$ and v_0 should everywhere be replaced by $v_0(\kappa)$ defined by Eq. (21).

κ	$Z_\kappa(\xi)$	z	$D(k,\omega)=1-(\omega_p^2/k^2v_0^2)Z'_\kappa(\xi=\omega/\lvert k\rvert v_0)$
1	$i(1-iz)^{-1}$	$z=\xi$	$1+(\omega_p^2/k^2v_0^2)(1-iz)^{-2}$
2	$(i/\sqrt{2})[(1-iz)^{-2}+(1-iz)^{-1}]$	$z=\xi/\sqrt{2}$	$1+(\omega_p^2/2k^2v_0^2)[2(1-iz)^{-3}+(1-iz)^{-2}]$
3	$(i/3\sqrt{3})[2(1-iz)^{-3}+3(1-iz)^{-2}+3(1-iz)^{-1}]$	$z=\xi/\sqrt{3}$	$1+(\omega_p^2/3k^2v_0^2)[2(1-iz)^{-4}+2(1-iz)^{-3}+(1-iz)^{-2}]$
4	$(i/10)[2(1-iz)^{-4}+4(1-iz)^{-3}$ $+5(1-iz)^{-2}+5(1-iz)^{-1}]$	$z=\xi/2$	$1+(\omega_p^2/20k^2v_0^2)[8(1-iz)^{-5}+12(1-iz)^{-4}$ $+10(1-iz)^{-3}+5(1-iz)^{-2}]$

$$f_\kappa(v) = \pi v_0^2 A_\kappa\left[1+\frac{v^2}{\kappa v_0^2}\right]^{-\kappa}, \quad \kappa > 3/2, \tag{23}$$

where $v=v_x$. It should be noted that the definition of v used here, $v=v_x$, is different from the definition of v used in the last section. The plasma dielectric function is obtained by substituting Eq. (23) into Eq. (14). This yields

$$D(k,\omega) = 1 - \frac{\omega_p^2}{k^2v_0^2}Z'_\kappa\left(\frac{\omega}{|k|v_0}\right), \tag{24}$$

where, for $\mathrm{Im}(\xi)>0$,

$$Z_\kappa(\xi) = \frac{1}{\sqrt{\pi\kappa}}\frac{\Gamma(\kappa)}{\Gamma(\kappa-1/2)}\int_{-\infty}^{\infty}\frac{1}{(v-\xi)(1+v^2/\kappa)^\kappa}dv \tag{25}$$

is the plasma dispersion function for the κ distribution and $Z'(\xi)=dZ/d\xi$. For $\mathrm{Im}(\xi)\leqslant 0$, the function $Z_\kappa(\xi)$ is defined by analytic continuation. In the limit as $\kappa\to\infty$, the function $Z_\kappa(\xi)$ approaches the limit

$$Z(\xi) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-v^2}}{v-\xi}dv, \quad \mathrm{Im}(\xi)>0, \tag{26}$$

which is the well-known plasma dispersion function for the Maxwell distribution. Thus the dispersion function (25) for the isotropic κ distribution generalizes the well-known result (26) for the isotropic Maxwell distribution.

The dispersion function (25) was derived by Hellberg and Mace[10] [see Eq. (10) in Ref. 9] who denote this function by $Z_{\kappa M}(\xi)$. Independently, the same function was derived in an unpublished work by Podesta[16] who denotes the function by $Z_\kappa(\xi)$. The latter notation shall be followed here. The plasma dielectric function for the κ distribution was given in the form (24) by Hellberg and Mace [see Eq. (61) in Ref. 9] and, independently, it was given in the same form by Podesta.[16] It should be noted that the plasma dispersion function (25) is different from the function $Z^*_\kappa(\xi)$ defined in an earlier work by Summers and Thorne,[17] a definition also adopted by Mace and Hellberg.[18] These pioneering papers employ an exponent $\kappa+1$ rather than κ in the integrand. Consequently, the function $Z^*_\kappa(\xi)$ of Summers and Thorne[17] is more closely related to the derivative $Z'_\kappa(\xi)$ of the function defined by Eq. (25). The definition (25) is convenient because it preserves the well-known form (24) for the plasma dielectric function.[14]

For integer values, $\kappa=n$, the integral (25) is readily evaluated using the calculus of residues with the result

$$Z_n(\xi) = i\frac{(n-1/2)}{n^{1/2}}\frac{\Gamma(n)}{\Gamma(2n)}\sum_{m=0}^{n-1}\frac{\Gamma(m+n)}{m!}\left(\frac{1-iz}{2}\right)^{m-n}, \tag{27}$$

where $z=\xi/\sqrt{n}$. The results for the first few values of n are listed in Table I. For arbitrary real values of the parameter κ, a simple way to evaluate the integral (25) is to first derive a differential equation for it and then express the solution in terms of a hypergeometric function. This is the approach developed by Podesta.[16] Another approach is to use the theory of contour integration. This is the original approach adopted by Mace and Hellberg.[18] Once it is expressed in terms of Gauss' hypergeometric function, the well-documented properties of the hypergeometric function can be applied to derive many useful properties of the plasma dispersion function. This program was carried out by Podesta.[16]

V. DISPERSION RELATION FOR SPATIAL DAMPING

The dielectric function (24) has been defined for k real and ω complex. To obtain the dispersion relation for the spatial damping problem it is necessary to extend the definition of $D(k,\omega)$ to the case where ω is real and k is complex. A description of this procedure is included here for completeness, however, the advanced reader may skip this section if desired.

The integral in Eq. (12) actually defines two different analytic functions depending on whether $k>0$ or $k<0$. This is because the integral is discontinuous as the complex variable $\xi=\omega/k$ crosses the real axis. For example, for the Maxwell distribution,

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-v^2}}{v-\xi}dv = \begin{cases} Z(\xi), & \mathrm{Im}(\xi)>0 \\ Z(\xi)-2\sqrt{\pi}ie^{-z^2}, & \mathrm{Im}(\xi)<0 \end{cases}, \tag{28}$$

where the function $Z(\xi)$ is defined throughout the complex plane by analytic continuation.

To take this into account, consider the function

$$D_+(k,\omega) = 1 - \frac{\omega_p^2}{k^2v_0^2}Z'_\kappa\left(\frac{\omega}{kv_0}\right), \tag{29}$$

where $k>0$ is real and positive, ω is any complex number, and, as always, $Z_\kappa(\xi)$ is defined throughout the complex plane by the analytic continuation of Eq. (25). If ω is a fixed real number, then Eq. (29) defines a unique analytic function of k which reduces to the function $D(k,\omega)$ defined in Eq.

(24) when k is real and positive. Likewise, consider the function

$$D_{-}(k,\omega) = 1 - \frac{\omega_p^2}{k^2 v_0^2} Z'_\kappa\left(\frac{-\omega}{k v_0}\right), \tag{30}$$

where $k<0$ is real and negative and ω is any complex number. If ω is a fixed real number, then this defines a unique analytic function of k which reduces to the function $D(k,\omega)$ in Eq. (24) when k is real and negative.

The two functions D_+ and D_- are different except that along the real axis $D_+(k,\omega)=D_-(-k,\omega)$, $k>0$, so that, as discussed previously, $D(k,\omega)$ in Eq. (24) is an even function of k. Moreover, from the definitions (29) and (30), it is clear that for any complex value of k the two functions are related by $D_+(k,\omega)=D_-(-k,\omega)$.

It is logical to define

$$D(k,\omega) = \begin{cases} D_+(k,\omega), & \mathrm{Re}(k) > 0 \\ D_-(k,\omega), & \mathrm{Re}(k) < 0 \end{cases}. \tag{31}$$

This yields a unique analytic continuation of $D(k,\omega)$ into the complex k plane such that the dispersion relation for spatial Landau damping takes the form $D(k,\omega)=0$. For $\omega>0$, the component D_+ is associated with waves that propagate from left to right and D_- is associated with waves that propagate from right to left. It is important to note, however, that the function defined by Eq. (31) is discontinuous across the imaginary axis in the complex k plane.

VI. SOLUTION FOR STATIC DEBYE SHIELDING

In this section it is shown that in the low frequency limit $\omega\to 0$ the solution of the boundary value problem defined by Eqs. (5) and (6) reduces to the well-known solution for static Debye shielding. By inspection, the solution (11) for the electric field is primarily determined by the properties of the plasma dielectric function (24). In the low frequency limit $\omega=0$, the relation

$$Z'_\kappa(0) = -\left(\frac{2\kappa-1}{\kappa}\right) \tag{32}$$

for the derivative of the plasma dispersion function[16] may be substituted into Eq. (24) to obtain

$$D(k,\omega=0) = 1 + \frac{\omega_p^2}{k^2 v_0^2}\left(\frac{2\kappa-1}{\kappa}\right) \equiv 1 + \frac{1}{(k\lambda_\kappa)^2}, \tag{33}$$

where, by definition, λ_κ is the Debye length.[19] Thus, the potential due to a static sheet charge decays like $\exp(-|x|/\lambda_\kappa)$ where the Debye length for a κ-distributed plasma is given by

$$\lambda_\kappa = \frac{v_0}{\omega_p}\left(\frac{\kappa}{2\kappa-1}\right)^{1/2} = \left(\frac{2\kappa-3}{2\kappa-1}\right)^{1/2}\lambda_D. \tag{34}$$

Here, the last equality on the right-hand side holds for an isothermal family of κ distributions and $\lambda_D=(\epsilon_0 k_B T/n_0 e^2)^{1/2}$ is the Debye length for the Maxwell distribution. In general, the Debye length for a κ distribution is less than but close to that of a Maxwell distribution as long as κ is not too close to $3/2$. For $\kappa=2$, one finds $\lambda_\kappa/\lambda_D \simeq 0.577$ with λ_κ/λ_D increasing to 1 as $\kappa\to\infty$. More information about the Debye length in κ-distributed plasmas can be found in the study by Bryant.[19]

VII. SEPARATION OF THE NEAR-FIELD AND FAR-FIELD COMPONENTS

If $|x|$ is very large, then the integrand in Eq. (15) is rapidly oscillating and the dominant contribution to the integral occurs near $k=0$. To see this, write the integral in Eq. (15) in the form

$$\int_0^\infty \frac{\sin(kx)}{kD(k,\omega)}dk = \mathrm{sgn}(x)\int_0^\infty \frac{\sin(u)}{uD(u/|x|,\omega)}du, \tag{35}$$

where $|x|$ is large and $\mathrm{sgn}(x)$ denotes the algebraic sign of x. In the neighborhood of $k=0$, the function $D(k,\omega)$ is approximately constant so that it can be removed from the integral with the result

$$\int_0^\infty \frac{\sin(u)}{uD(u/|x|,\omega)}du \simeq \frac{\pi}{2D(0,\omega)}. \tag{36}$$

Using the asymptotic series or the closed form expressions for the plasma dispersion function, it follows that, for real values of k,

$$D(k,\omega) \simeq 1 - \frac{\omega_p^2}{\omega^2} \quad \text{as } k\to 0. \tag{37}$$

Therefore, the far-field component of the electric field (15) is given by

$$\tilde{E}_F(x,\omega) = \frac{\rho_0}{2\epsilon_0[1-(\omega_p^2/\omega^2)]}\,\mathrm{sgn}(x). \tag{38}$$

In the absence of the plasma, the electric field due to the sheet charge is $(\rho_0/2\epsilon_0)\mathrm{sgn}(x)$. In the presence of the plasma the electric field at large distances (38) is modified by the long-wave-number dielectric function $1-(\omega_p^2/\omega^2)$. This behavior was found by Landau[12] in his solution of the spatial damping problem and also by Buckley[15] in his solution $e(x)$ of the dynamic screening problem.

Writing the total electric field (11) in the form $E=E_N+E_F$, one finds for the near-field component

$$\tilde{E}_N(k,\omega) = \frac{\rho_0}{ik\epsilon_0}\left[\frac{1}{D(k,\omega)} - \frac{1}{1-(\omega_p^2/\omega^2)}\right]. \tag{39}$$

The near field has the important property that it is regular at $k=0$, that is, it behaves like a positive integral power of k as $k\to 0$ and, therefore, the electrostatic potential $\tilde{\phi}(k,\omega)=-\tilde{E}_N(k,\omega)/ik$ is well behaved near $k=0$. As a consequence, the potential vanishes as $|x|\to\infty$. To prove this, write the inverse Fourier transform of the potential $\tilde{\phi}(k,\omega)$ as a cosine transform, which is possible since $D(k,\omega)$ is an even function of k, and then integrate once by parts.

As discussed in Sec. II, the plasma waves generated by an electrode with an oscillating potential are described by the near-field solution (39). Hence, the desired solution of the original boundary value problem is given by

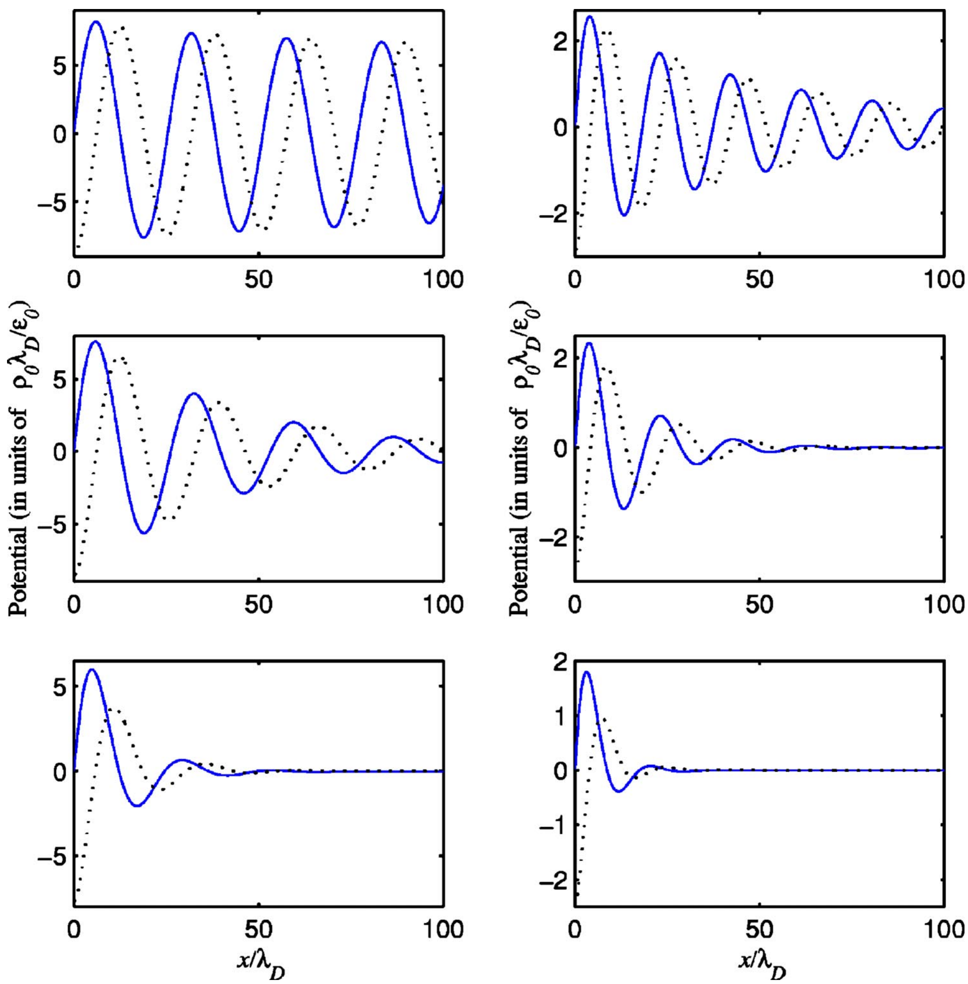

FIG. 1. Solution for the in-phase (solid) and quadrature components (dotted) of the potential $\phi(x,t)$ computed from Eq. (41) at the frequencies $\omega/\omega_p=1.1$ (left) and $\omega/\omega_p=1.2$ (right). From top to bottom, results are shown for κ-distributed plasmas with parameters $\kappa=\infty$ (Maxwell distribution), $\kappa=4$, and $\kappa=2$, respectively.

$$\tilde{\phi}(k,\omega)=\frac{\rho_0}{k^2\epsilon_0}\left[\frac{1}{D(k,\omega)}-\frac{1}{1-(\omega_p^2/\omega^2)}\right], \tag{40}$$

where $D(k,\omega)$ is given by Eq. (24). If the removal of the far field somehow seems arbitrary, note that, in practice, one may eliminate the far field by introducing a second electrode with an equal and opposite charge, thereby forming a capacitor. The field exterior to the capacitor is then given by Eq. (40). The inverse Fourier transform of Eq. (40) yields

$$\tilde{\phi}(x,\omega)=\frac{\rho_0}{\pi\epsilon_0}\int_0^\infty\left[\frac{1}{D(k,\omega)}-\frac{1}{1-(\omega_p^2/\omega^2)}\right]\frac{\cos(kx)}{k^2}dk, \tag{41}$$

where the exact expression for $D(k,\omega)$ is given by Eq. (24). This integral converges because the quantity in square brackets obeys the asymptotic relations $[\cdots]\sim k^2$ as $k\to 0$ and $[\cdots]\sim 1$ as $k\to\infty$. In the following section, the Fourier integral (41) is computed numerically.

VIII. SOLUTIONS FOR THE ELECTROSTATIC POTENTIAL

In general, the solutions behave differently depending on whether the driving frequency is greater or less than the plasma frequency. Below the plasma frequency, when $\omega/\omega_p<1$, the solutions are rapidly damped within a few Debye lengths of the source. These solutions are of no interest here and are discussed in detail by Buckley.[15] When $\omega/\omega_p>1$, electrostatic plasma waves are excited in the plane $x=0$ and undergo spatial Landau damping as they propagate away from the source. The damping increases as the parameter ω/ω_p is increased. As $\omega/\omega_p\to 1^+$, the wavelength approaches infinity and the damping approaches zero; this is the long-wavelength limit.

The exact solution for the potential $\phi(x,t)$ is obtained by computing the inverse Fourier transform of Eq. (40) using the fast Fourier transform (FFT) algorithm. The dielectric function $D(k,\omega)$ is computed using the closed form expressions in Table I or, for the Maxwell distribution, by using the very accurate and efficient numerical algorithm developed by Gautschi.[20] All calculations are performed using the dimensionless variables $\omega'=\omega/\omega_p$ and $k'=k\lambda_D$ where the Debye length λ_D is defined in Sec. VI. In addition, all the distribution functions are characterized by the same absolute temperature T. The results for the in-phase and quadrature components of $\phi(x,t)$ are plotted in Fig. 1 for the cases $\omega/\omega_p=1.1$ and $\omega/\omega_p=1.2$. The solutions are even functions of x so only the region $x>0$ is shown in the figure.

The results shown in Fig. 1 clearly illustrate the dependence of the damping on κ. As κ decreases, the particle population in the high energy tails increases and, consequently, the damping increases because there are more reso-

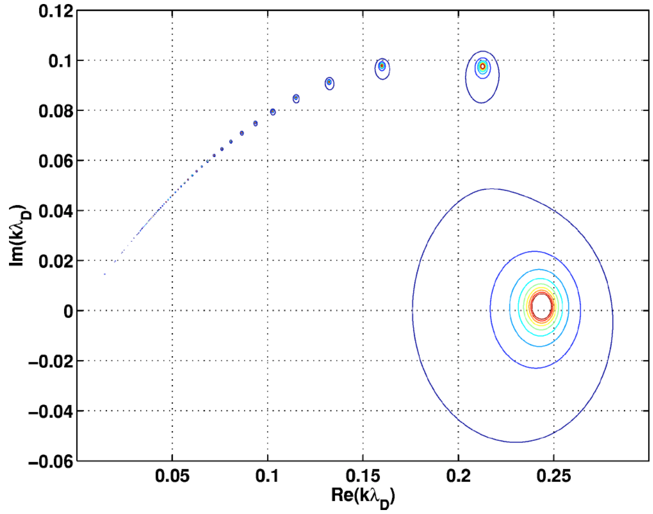

FIG. 2. Contour plot showing the poles of $1/|D_+(k,\omega)|$ for the Maxwellian dispersion relation in the case $\omega/\omega_{pe}=1.1$. The dominant pole is the one furthest to the right. A pole at $k\lambda_D \simeq 1.62i$ having a width comparable to the dominant pole is a purely damped mode and is not shown on this plot.

nant particles available to participate in the damping process. Landau damping is strongest for small values of κ, such as $\kappa=2$, 3, or 4. For moderate to large values of κ, the damping approaches the thermal equilibrium value. In space plasmas, such as the solar wind and the magnetosphere,[1,2,21–23] observations typically yield values ranging from $\kappa=3$ to $\kappa=6$.

IX. APPROXIMATE SOLUTIONS

It is of interest to compare the exact solutions discussed in the last section with the solutions suggested by the theory of normal modes, namely,

$$\tilde{E}_N^{\text{approx}}(x,\omega) = E_0 \exp[i(k_r + ik_i)|x|]\text{sgn}(x) \tag{42}$$

and

$$\tilde{\phi}^{\text{approx}}(x,\omega) = \phi_0 \exp[i(k_r + ik_i)|x|], \tag{43}$$

where E_0 and ϕ_0 are constants and $k(\omega)=k_r+ik_i$, with $k_r>0$ and $k_i>0$, is the dominant root of the dispersion relation. For a κ distribution with integer values of κ there exists a unique root of the dispersion relation with the smallest imaginary part, the least damped root, which is the "dominant root" in this case. For the Maxwell distribution the dispersion relation possesses an infinite number of roots with decreasing imaginary part so there is no "least damped" root in this case. A contour plot showing the locations of the roots in the first quadrant is shown in Fig. 2. The dominant root in this case is the one furthest to the right in Fig. 2 and is such that the function $1/|D(k,\omega)|$ has a pole at this point with a very broad peak. The other roots have poles with successively narrower peaks. The fact that the solution of the spatial damping problem is dominated by this one root has been noted previously by Gould[24] and by Buckley.[15]

The values of the coefficients E_0 and ϕ_0 in Eqs. (42) and (43) cannot be determined from the dispersion relation, however, they can be estimated from the exact solutions (15) and (41). The limit of Eq. (15) as $x\to 0$ is obtained by using the limit $D(k,\omega)\to 1$ as $k\to\infty$, which implies

$$\tilde{E}(x,\omega) = \frac{\rho_0}{2\epsilon_0}\,\text{sgn}(x) \quad \text{as } x \to 0. \tag{44}$$

Subtracting the far-field (38) from the total field (44), one obtains the near-field

$$\tilde{E}_N(x,\omega) = -\frac{\rho_0}{2\epsilon_0}\left(\frac{\omega_p^2}{\omega^2-\omega_p^2}\right)\text{sgn}(x) \quad \text{as } x \to 0, \tag{45}$$

which implies

$$E_0 = -\frac{\rho_0}{2\epsilon_0}\left(\frac{\omega_p^2}{\omega^2-\omega_p^2}\right). \tag{46}$$

Now proceed to the evaluation of ϕ_0. In the limit as $x\to 0$, Eq. (41) yields

$$\tilde{\phi}(x=0,\omega) = \frac{\rho_0}{\pi\epsilon_0}\int_0^\infty \left[\frac{1}{D(k,\omega)} - \frac{1}{1-(\omega_p^2/\omega^2)}\right]\frac{dk}{k^2}. \tag{47}$$

This integral must be computed numerically. For reasons that are not immediately obvious, the real part of Eq. (47) is zero, at least, for the cases considered here. As a consequence, the potential and the electric field are out of phase by $\pi/2$ rad. Using a standard numerical integration routine to evaluate the integral in Eq. (47), one finds that for $\omega/\omega_p=1.1$,

$$\phi_0 \simeq \frac{\rho_0\lambda_D}{\epsilon_0} \times \begin{cases} -7.65i & \text{for } \kappa=2 \\ -8.56i & \text{for } \kappa=4 \\ -8.75i & \text{for } \kappa=\infty. \end{cases} \tag{48}$$

Substituting these values into Eq. (43), multiplying by $e^{-i\omega t}$, and then taking the real part, one finds

$$\phi^{\text{approx}}(x,t) = |\phi_0|\sin(k_r|x| - \omega t)\exp(-k_i|x|). \tag{49}$$

To complete the solution it is necessary to find the roots of the dispersion relation.

For $\kappa=2$, the dispersion relation from Table I may be written as

$$1 - \frac{\omega_p^2}{\omega^2}(-iz)^2[2(1-iz)^{-3} + (1-iz)^{-2}] = 0, \tag{50}$$

where $z=\omega/\sqrt{2}kv_0$ or, equivalently, $z=(\omega/\omega_p)/k\lambda_D$. By omitting the absolute value signs on k, this becomes the equation $D_+(k,\omega)=0$ discussed in Sec. V. Making the change of variable $z=iy$ followed by $x=(1+y)^{-1}$, one obtains the equation

$$2x^3 - 3x^2 = \frac{\omega^2}{\omega_p^2} - 1. \tag{51}$$

If $\omega>\omega_p$, then, by Descarte's rule of signs, this equation has one positive real root. Using Newtons method with the initial guess $x=3/2$, one finds, for $\omega/\omega_p=1.1$, the root $x=1.544\,04\ldots$ Using this root to factor the polynomial (51), the remaining roots are then given by the quadratic formula with the result $k\lambda_D=\pm 0.257+0.0890i$.

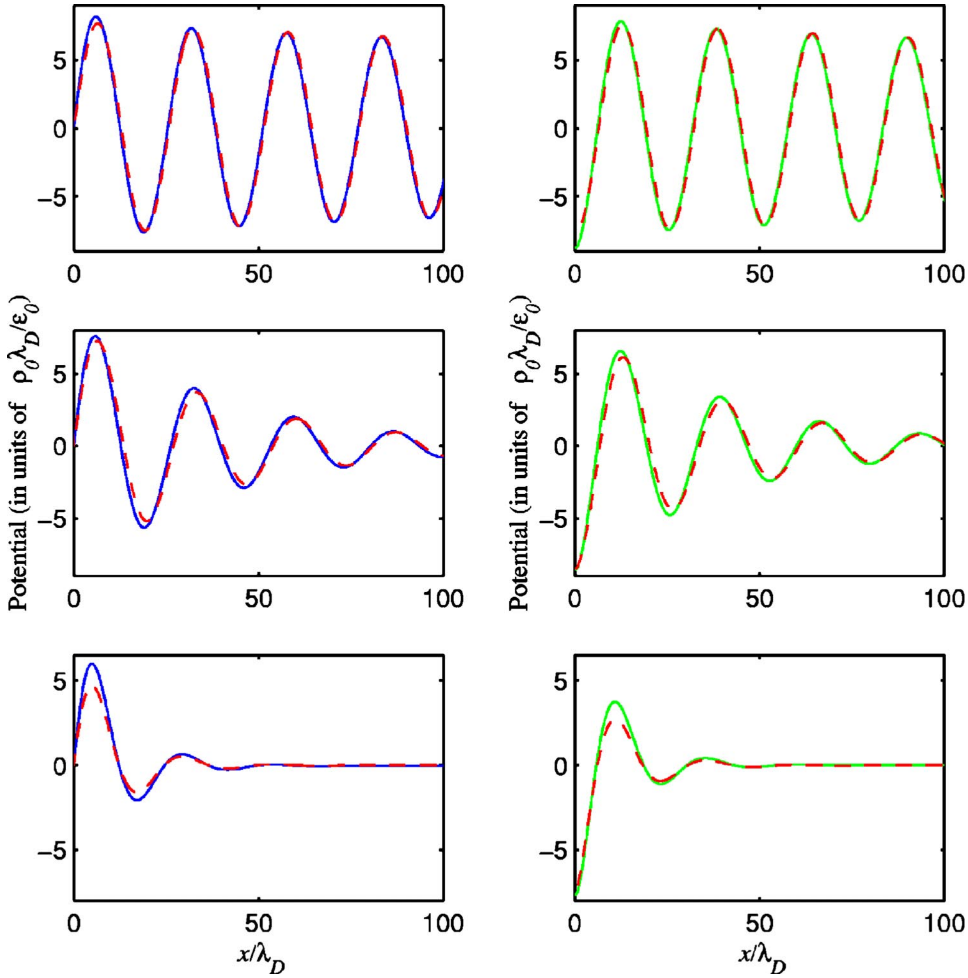

FIG. 3. Comparison between the exact solutions (solid) and the approximate solutions (dashed) for the in-phase (left column) and quadrature components (right column) of the potential $\phi(x,t)$. From top to bottom, results are shown for κ-distributed plasmas with parameters $\kappa=\infty$ (Maxwell distribution), $\kappa=4$, and $\kappa=2$, respectively. The driving frequency is $\omega/\omega_p=1.1$.

For $\kappa=4$, the dispersion relation from Table I may be written as

$$1+\frac{\omega_p^2}{5\omega^2}z^2[8(1-iz)^{-5}+12(1-iz)^{-4}+10(1-iz)^{-3} + 5(1-iz)^{-2}]=0, \tag{52}$$

where $z=\omega/2kv_0$ or, equivalently, $z=(\omega/\omega_p)/\sqrt{5}k\lambda_D$. The roots are found using Newton's method once an approximate root is obtained by first plotting the function $1/|D(k,\omega)|$ in the complex k plane. For $\omega/\omega_p=1.1$, this procedure yields the root $k\lambda_D=0.233+i0.0251$.

For $\kappa=\infty$, the dispersion relation for the Maxwell distribution may be written as

$$D(k,\omega)=1-\frac{\omega_p^2}{\omega^2}\xi^2 Z'(\xi)=0, \tag{53}$$

where $\xi=\omega/kv_0$ or, equivalently, $\xi=(\omega/\omega_p)/\sqrt{2}k\lambda_D$. The partial derivative of D with respect to k is computed using the differentation formula for $Z(\xi)$ and then the roots of (53) are computed using Newton's method. An approximate root is obtained by first plotting the function $1/|D(k,\omega)|$ in the complex k plane. For $\omega/\omega_p=1.1$, this procedure yields the root $k\lambda_D=0.244+i0.001\,68$.

Using the roots of the dispersion relation just obtained, the approximate solutions (49) at the driving frequency $\omega/\omega_p=1.1$ take the form

$$\phi(x,t)\simeq\frac{\rho_0}{\epsilon_0\lambda_D}(7.65)\sin(0.257|x|-\omega t)\exp(-0.089|x|) \tag{54}$$

for $\kappa=2$,

$$\phi(x,t)\simeq\frac{\rho_0}{\epsilon_0\lambda_D}(8.56)\sin(0.233|x|-\omega t)\exp(-0.025|x|) \tag{55}$$

for $\kappa=4$, and

$$\phi(x,t)\simeq\frac{\rho_0}{\epsilon_0\lambda_D}(7.75)\sin(0.244|x|-\omega t)\exp(-0.00168|x|) \tag{56}$$

for $\kappa=\infty$. Here, the distance x is in units of λ_D. The exact solutions obtained from Eq. (41) and the approximate solutions (54)–(56) are compared in Fig. 3. As can be seen from Fig. 3, the normal mode solutions provide a reasonably good fit to the exact solutions. It should be mentioned that the amplitude in Eq. (56) has been reduced from 8.75, the true value of the potential at $x=0$, to the value 7.75 in order to give a better fit to the exact solution. This adjustment is necessary because for the Maxwell distribution there is a rapidly damped component of the solution which is not included in the one term approximation (49). It should be kept in mind that the goal of the normal mode approximation is only to obtain the real and imaginary parts of the wave number, not the wave amplitude (the amplitude cannot be inferred from the dispersion relation). With this caveat, the results of this section show that the normal mode solutions yield a reasonably good approximation to the exact solutions.

X. DISCUSSION

The boundary value problem for a transparent grid electrode with an oscillating sheet charge is closely related to the boundary value problem studied by Gould.[24] Gould studied the excitation of plasma waves by a pair of two closely spaced grid electrodes, a dipole layer. It is interesting that the electrostatic potential due to an oscillating dipole layer is identical to the electric field of an oscillating sheet charge. Therefore, the boundary value problem studied in this paper is similar in many respects to the problem studied by Gould.

Consider a dipole layer with charge density

$$\tilde{\rho}(x,\omega) = \frac{\sigma_0}{2}[\delta(x+h) - \delta(x-h)], \tag{57}$$

where $\sigma_0/2$ is the surface charge density and the distance h between the electrodes is small. The Fourier transform yields

$$\tilde{\rho}(k,\omega) = i\sigma_0 \sin(kh). \tag{58}$$

In the limit $h \to 0$, $\sigma_0 \to \infty$, $\sigma_0 h \to \rho_0 = \text{const}$, Eq. (58) becomes $\tilde{\rho}(k,\omega) = ik\rho_0$ or, equivalently,

$$\tilde{\rho}(x,\omega) = \rho_0 \delta'(x). \tag{59}$$

This should be compared to the charge density in Eq. (6), namely,

$$\tilde{\rho}(x,\omega) = \rho_0 \delta(x). \tag{60}$$

Therefore, if one makes the substitutions $\tilde{E} \to -ik\tilde{\phi}$ and $\rho_0 \to ik\rho_0$ in Eq. (11), it follows that the solution for the electric field due to an oscillating sheet charge and the solution for the potential due to a dipole layer are equivalent, except for a minus sign.

The important difference between the two solutions is that the potential derived in Sec. VIII is an even function of x whereas the potential in Gould's problem is an odd function of x. Consequently, the solution of Gould's problem is not an appropriate model for the electrostatic waves generated by a plane electrode with an applied potential since, by physical considerations, the correct solution must have even symmetry.

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APPENDIX: DERIVATION OF EQ. (15)

In this appendix it is shown that the inverse Fourier transform of Eq. (11) is given by Eq. (15). Consider the function $1/k$ which has the inverse Fourier transform $\text{sgn}(x)/2$. This Fourier transform pair is denoted by

$$\frac{1}{2}\,\text{sgn}(x) \leftrightarrow \frac{1}{k}. \tag{A1}$$

The inverse Fourier transform of $1/k$ cannot be computed using the definition of the Fourier transform in terms of an integral. It only has meaning in the sense of the theory of distributions (see, for example, Ref. 25, Chap. IX). Accepting the correctness of Eq. (A1), now write Eq. (11) in the form

$$\frac{1}{ikD(k)} = \frac{1}{ik}\left[\frac{1}{D(k)} - \frac{1}{D(0)}\right] + \frac{1}{ikD(0)} \tag{A2}$$

where the dependence on ω has been omitted to focus attention on the k dependence. Applying the Fourier transform operator $\mathcal{F}$ to both sides of this equation and then using the linearity property, one obtains

$$\mathcal{F}^{-1}\left[\frac{1}{ikD(k)}\right] = \mathcal{F}^{-1}\left\{\frac{1}{ik}\left[\frac{1}{D(k)} - \frac{1}{D(0)}\right]\right\} + \mathcal{F}^{-1}\left[\frac{1}{ikD(0)}\right]. \tag{A3}$$

As indicated in the discussion following Eq. (39), the first term on the right-hand side of Eq. (A3) contains no singularities and can be defined as an ordinary integral over k. The singularity is contained in the second term on the right-hand side which has an inverse Fourier transform given by Eq. (A1). Thus, the first and second terms on the right-hand side of Eq. (A3) are equivalent to the near-field and far-field components denoted by $\tilde{E}_N$ and $\tilde{E}_F$ in Sec. VII, respectively.

Independently, it is shown in Sec. VII that the integral in Eq. (15) is equal to the sum of $\tilde{E}_N$ and $\tilde{E}_F$. Therefore, the inverse Fourier transform in Eq. (A3) must be equal to the integral in Eq. (15) This proves the desired result. In short, according to the theory of distributions, the singularity which occurs in the inverse Fourier transform of Eq. (11) is handled correctly by using the Cauchy principle value to evaluate the integral. The same recipe also produces the correct inverse Fourier transform of $1/k$ given in Eq. (A1).

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